Prove It: Investigating Mathematical Proofs as a Writing Genre

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Too many people believe that their future careers won't involve any more writing than is necessary to fill out a resume. Jonathan Sabin reveals this notion to be nothing more than wishful thinking in his article about why even something as seemingly antithetical to English as math requires the ability to communicate clearly and effectively through writing.

No, you didn't misread the title. This article is about mathematical proofs. And yes, mathematical proofs are a genre of writing. This may come as a surprise to many readers. After all, most people reading this will not have dealt with proofs since they were underclassmen in high school. Some may not even remember what a proof is. Those who do remember will probably be confused by my assertion of proofs as a writing genre, but it's true. A formal mathematical proof requires just as much writing ability as mathematical ability, sometimes even more so. That is why I have chosen to write this in-depth analysis of proofs as a genre. And yes, my good friends: this means that CHAT, our helpful and reliable—if somewhat annoyingly omnipresent—companion will be coming along for the ride.

I say "annoying," but, in all honesty, I make fun of that which I love because CHAT is one of our most helpful tools in understanding exactly why proofs constitute a genre of writing. In her article, "Just CHATting," Joyce R. Walker explains that ISU's version of CHAT, or cultural-historical activity theory, "refers to a set of theories about rhetorical activity . . . that help us look at the how/why/what of writing practices" (Walker, 2010, pp.

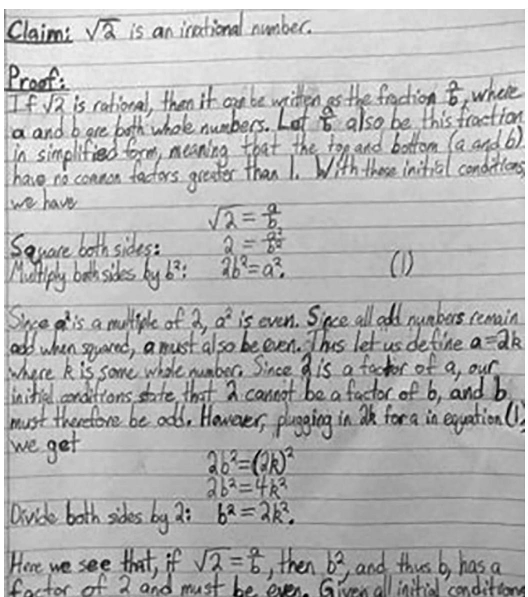
71–72). In the ISU Writing Program, these theories center around seven different terms that help ISU students analyze literate activity and outline the different aspects that factor into how a person goes about writing a text. We will not cover all seven of these literate activities, but we will come across a few of them in our discussion.

Before we dive into the rest of this article, however, there is a matter that I wish to briefly address. Although the opportunity for me to write a piece for this journal initially spawned from a writing assignment I completed for an English class, I did not choose to accept the offer simply because my professor wanted me to. It is true that I wrote this article, in part, to explore the full implications of the classification of proofs as a writing genre and to explain the reasoning behind it. However, there is also a far more important and meaningful message behind all this: it is a lesson that I believe ought to be conveyed far more often, especially to college freshmen. It is one that I have reason to wish had been taught to me much sooner. But we'll come back to that later. First, a quick refresher.

What Is a Proof?

A mathematical proof is a process by which a chain of logic is presented and followed through to completion in order to show conclusively and indisputably that a specific mathematical claim will always be true. For any new mathematical idea to be taken as fact, it must first be mathematically proven. After all, math is the medium with which we quantify and analyze the world around us. "We need math," says Dale Stokdyk, the Assistant Vice President of Southern New Hampshire University, in his article titled "Importance of Mathematics and Why We Study It:" "Galileo Galilei used it to explain the universe. Math resolves truths and uncovers errors. It makes our work more credible. Reports, studies and research are all but discounted without quantifiable facts" (Stokdyk, 2016). For our understanding of the world to be accurate, therefore, mathematics and, by extension, all mathematical concepts, must be provably accurate. It is for this reason that a complete understanding of mathematics is impossible without an understanding of proofs.

As an example of this, let us draw upon the "historical" aspect of the CHAT acronym and consider the ancient past. Around the fifth century B.C., there was a group of mathematicians called Pythagoreans, named after the famed mathematician Pythagoras. These mathematicians believed that every number in existence was expressible as a ratio of two whole numbers—that is, that every number was equal to one whole number divided by another whole number, a numerical property referred to as rationality. This was the Pythagoreans' most strongly held mathematical conviction. Indeed, the strength of this belief was likely bolstered by the fact that, to them, it had religious significance; the Pythagoreans, in the words of researcher and science writer Brian Clegg, in his online article titled "The Dangerous Ratio," "believed that the universe was built around the whole numbers" (Clegg, 2011). So, imagine their shock when they turned out to be wrong. History is a bit fuzzy on who exactly it was that proved that not all numbers were rational, but the popular belief is that it was the Pythagorean mathematician named Hippasus of Metapontum. So, for the sake of simplicity, we'll just assume that it was he who first discovered the existence of irrational numbers. How did he do this? He formulated a proof demonstrating that the square root of two was irrational. I have included an image of this proof below (Figure 1). The technical aspects of this proof are not important for the purposes of this article, but those who wish to view a very easily followed explanation of the reasoning can visit the URL listed in the first entry on the Works Cited page of the article. For our purposes, it



Claim: $\sqrt{2}$ is an irrational number.

Proof:
If $\sqrt{2}$ is rational, then it can be written as the fraction $\frac{a}{b}$, where a and b are both whole numbers. Let $\frac{a}{b}$ also be this fraction in simplified form, meaning that the top and bottom (a and b) have no common factors greater than 1. With these initial conditions, we have

$$\sqrt{2} = \frac{a}{b}$$

Square both sides: $2 = \frac{a^2}{b^2}$

Multiply both sides by b^2: $2b^2 = a^2$. (1)

Since a^2 is a multiple of 2, a^2 is even. Since all odd numbers remain odd when squared, a must also be even. Thus let us define $a = 2k$ where k is some whole number. Since 2 is a factor of a, our initial conditions state that 2 cannot be a factor of b, and b must therefore be odd. However, plugging in 2k for a in equation (1), we get

$$2b^2 = (2k)^2$$
$$2b^2 = 4k^2$$

Divide both sides by 2: $b^2 = 2k^2$.

Here we see that, if $\sqrt{2} = \frac{a}{b}$, then b^2, and thus b, has a factor of 2 and must be even. Given all initial conditions b must be both even and odd, which is impossible. Therefore the initial conditions for rationality are not met, and $\sqrt{2}$ is irrational.

Figure 1: A variation on the most commonly taught mathematical proof that the square root of two is an irrational number. I wrote it out by hand with the intention of making the logic of the proof easier to follow.

suffices to say that Hippasus discovered that any fraction of whole numbers that equaled the square root of two would have to have a denominator that was simultaneously even and odd, which is an impossibility.

Now, to the non-mathematically-inclined reader, this discovery may seem insignificant. But be assured that it was a discovery of massive consequence and implications with regards to our understanding of mathematics. So much so, in fact, that I would be utterly remiss if I didn't take this opportunity to discuss reception, one of the seven terms of ISU CHAT. Reception, according to Walker, "deals with how a text is taken up and used by others" (Walker, p. 75). The reception of written proofs has to do with the mathematical community's understanding of the underlying mathematical concept as well as how it affects the landscape of mathematical knowledge. This particular proof would send shockwaves through that landscape, forever changing our perception of mathematics. It demonstrated that a large portion of mathematical knowledge up to that point was founded upon an incorrect notion. It certainly shocked the Pythagoreans. So much so that, according to popular legend, they became rather irrational themselves and drowned Hippasus at sea for his heretical discovery. Whether or not this last part is really true is, at this point, neither certain nor important. But it's amusing to imagine a group of disillusioned mathematicians ganging up on one of their own and murdering him over a math problem. Entertaining legends aside, I hope the reader now appreciates the importance and potential significance of a mathematical proof. Every new proof further shapes and develops our overall understanding of the field of mathematics.

The Problem of Teaching and Learning Proofs

Despite this fact, student math-lovers from all over struggle with proofs. The fact that such an important part of mathematics is so difficult for so many students to grasp, even for those most proficient in math, has resulted in a good deal of frustration for, and even outright hatred of, mathematical proofs. So, the question, then, is what exactly makes proofs so hard for so many students?

In my search for an answer to this question, I looked to the students themselves. This was not a difficult thing for me to do. I simply Googled the question: "Why do people hate proofs?" The search results are full of forums in which primarily high school students air their grievances on the subject. Many claim that their teachers are to blame. The reason for this is explained in great detail in an article by Zacharie Mbaïtiga titled "Why

College or University Students Hate Proofs in Mathematics?" The subject of proofs is, after all, a fairly difficult one to teach, as most theorems in math are proved using other, already-proven theorems. It is not necessary for a proof to contain additional proofs for all these already-proven theorems, so most don't. This means that many of the proofs used as examples by teachers involve other concepts that their students aren't familiar with and that the teacher doesn't take the time to prove or even explain (Mbaïtiga, 2009, p. 35). This problem has much to do with the ISU CHAT term, socialization, which Walker says "describes the interactions of people and institutions as they produce, distribute and use texts" (Walker, p. 76). Among math educators, there appears to be difficulty when it comes to engaging effectively in their interaction with students when it comes to teaching them about proofs.

Another reason for the animosity towards proofs has to do with their purpose. What are they for? Are they necessary? Many of the students on these forums seem to think that they aren't, some even venting their frustration on this point very aggressively. For example, one of the comments in one forum thread was that of a student explaining why he "knows" that proofs are stupid. I unfortunately will not be quoting said comment in this article for the simple reason that it is unprintable in this journal. Suffice it to say that many students feel offended that, in a subject in which they know they excel, there is a seemingly unnecessary concept that they absolutely cannot understand, however hard they try. And this reaction is in many ways understandable, but the fact of the matter is that proofs are one of the most *absolutely* necessary parts of mathematics. Without proofs, nothing in mathematics would have any credibility, and the whole discipline would basically amount to highly sophisticated guesswork.

Writing About Math

OK, so perhaps at this point the reader is thinking, "Why is there an article about math proofs in a journal about writing? What does math have to do with English?" The answer is quite simple. While it is true that the formulation of proofs doesn't involve much writing, once the proof is complete it *does* need to be written, documented, and published. It also needs to be properly explained, and such an adequate explanation must be in words. After all, as I myself had not learned until I took this deeper look at proofs, mathematics is the quantitative, numerical, and analytical representation of thoughts and ideas, which are always conveyed through language. It is therefore imperative that mathematical proofs be treated less as a type of math problem and more as a specialized writing genre.

Interview

For further elaboration on mathematical proofs as a type of written genre, as well as further discussion regarding their necessity in the field of mathematics, I have included a portion of an interview, conducted some weeks prior as part of a separate writing project, with Sunil Chebolu, PhD, a professor of mathematics here at Illinois State University.

JONATHAN: Can you explain in general terms why mathematical proofs are so important? If you were explaining it to someone who wasn't as familiar with the field of mathematics why proofs are essential?

SUNIL: Yes, mathematical proofs are essential, because if you want to know something with certainty, how can you do that without a mathematical proof? With computers, by hand, you can only check finitely many things, but if you want to know that a statement holds for every integer or is true in all cases, you can't possibly do it all by hand or computer. Using a computer or by hand, you can say, "This is most likely true because it is true for a wide range of examples," but you have to make sure that it holds for all.

J: And in your career as a mathematician, how often did you have to formulate mathematical proofs? How often did you have to use them?

S: Oh, almost every day! That's like the bread and butter for mathematicians. When I'm teaching courses I prove things, when I'm doing research I prove things, publishing papers, giving talks. Yes, proofs are everywhere for us.

J: So aside from the mathematical processes involved, what else goes into writing a proof? Because I've seen a lot of proofs that required more than just the correct application of numbers and operators. Many of them even used more verbal writing than mathematical writing.

S: Oh, yeah. In fact, in most proofs, it's mostly words. I mean, people have this misconception that mathematical proof is all numbers and equations. No. Mathematical proof is about ideas. So numbers, equations, formulas are all representations of some ideas, right? So mathematical proof is just an assembly of ideas in a clear and coherent manner that takes you from the hypothesis to the conclusion. So that's how I think about mathematical proofs.

J: Yeah. And obviously, not all those ideas can be explained in just numbers—

S: Right.

J: Now, in my research on this subject, I learned that writing proofs isn't something that's really done in mathematical professions outside of academics and that it's mainly only academic mathematicians who do that kind of work. So how then are those proofs used by the mathematical community at large?

S: Well, I don't think the mathematical community at large, outside pure mathematics, really uses those proofs. They use more the results that we prove. Because, for instance, applied mathematics or business; there it's more computation, rather than proofs. So we develop the techniques and results, and they take those end products and use them in their work. They don't really worry about how that theorem is proved because we already do that, and we publish those results.

J: So proof writing is more a case of laying the groundwork for the mathematical ideas and tools used by the mathematical community–

S: To justify why those things are correct. Because if you want to use something in your work, you'd better know it's correct. How do you know it's correct? You need a proof. And that is what pure mathematicians do.

I feel it is important that I repeat a key point stated within the interview, one which I mentioned earlier. Chebolu described mathematical notation as a representation of mathematical ideas. This is an especially important point because the way in which people share and communicate thoughts and ideas is through the medium of spoken and written language. Therefore, the communication of the mathematical ideas represented by all those numbers and equations must be carried out through that same medium of language. This is why quality writing is so important in a mathematical proof. However, it would be very naïve of me to claim that the writing ability required for a proof is the same kind of writing ability displayed in an essay or this article, for example. Which is why I took a deeper look at proofs from a compositional standpoint.

Writing a Proof

As I carefully considered how best to perform the task of writing about this particular genre, it occurred to me that I'd never actually written a formal mathematical proof before. As a mathematics major, I had of course formulated proofs for homework assignments, quizzes, and exams, but I'd never later constructed them into complete, fully written-out, formal proofs. It dawned on me that, if I presumed to be seen as any kind of credible writer on this subject, I would have to construct a formal proof from scratch and derive my own insight thereby. And that's exactly what I did. That proof can be found in the appendix of this article for anyone who may be interested.

Before I continue, I feel it is important to briefly outline the difference between a formal and informal proof. A formal proof, such as the one shown in the appendix, is a proof that is properly typed, styled, and formatted in the standard, approved manner for mathematical publication. An informal proof is a shortened version of the formal proof, usually handwritten and containing many symbols that serve as mathematical shorthand for phrases such as “therefore,” “for any,” “there exists,” “such that,” etc. Non-mathematicians are not expected to know the meanings of these symbols, so formal proofs are expected to contain written explanations instead, making them much longer yet far easier to understand. The reader will therefore be forgiven for finding the informal proof below (Figure 2) impossible to follow. The far more coherent proof in the appendix is the formal version of the same proof.

Now, in my defense (just in case any of my math professors get to read this), it was the first time I had ever even thought to write a formal proof. But the point of doing so wasn’t to become an expert. I wrote this proof to gain an insight into proof writing that the experts, who hardly have to think about the process of writing proofs anymore because they’re so used to doing it, might no longer even consider. This being the primary focus of this exercise, I compiled a list of observations (Figure 3) relevant to that purpose as I went about writing the proof.

The most difficult part was presenting mathematical ideas in written form, and, as it turned out, I was attempting to do a good deal too much of this. I had originally planned to explain every mathematical term in the proof as to a layman (see Figure 4 below for a picture of my first attempted draft). As a result, my first attempt is so bogged down in the explanations of things like notation and definitions that it proved counterintuitive.

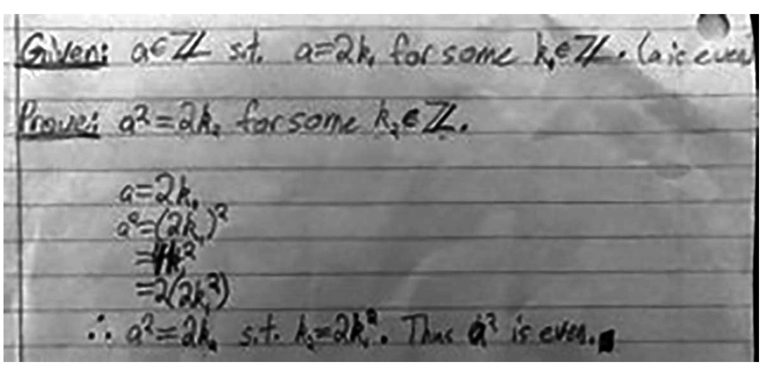
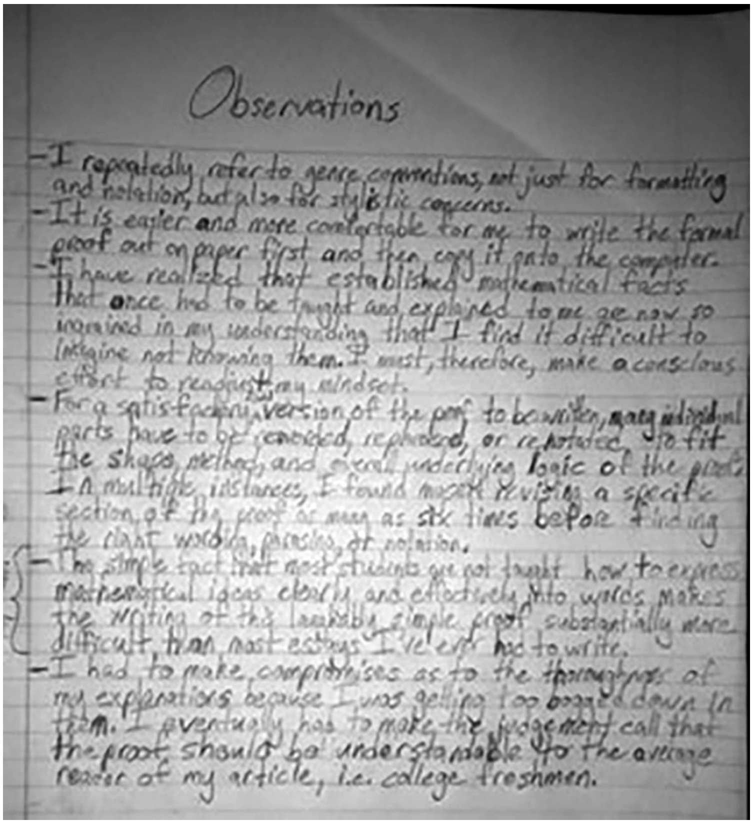


Figure 2: A handwritten proof that is the informal counterpart to the formal proof in the appendix of the article. Note the abundance of shorthand symbols within the text of the proof.


Observations

- I repeatedly refer to genre conventions, not just for formatting and notation, but also for stylistic concerns.
- It is easier and more comfortable for me to write the formal proof out on paper first and then copy it onto the computer.
- I have realized that established mathematical facts that once had to be taught and explained to me are now so ingrained in my understanding that I find it difficult to imagine not knowing them. I must, therefore, make a conscious effort to readjust my mindset.
- For a satisfactory version of the proof to be written, many individual parts have to be reworded, rephrased, or renotated to fit the shape, method, and overall underlying logic of the proof. In multiple instances, I found myself revising a specific section of the proof as many as six times before finding the right wording, phrasing, or notation.
- The simple fact that most students are not taught how to express mathematical ideas clearly and effectively into words makes the writing of this laughably simple proof substantially more difficult than most essays I've ever had to write.
- I had to make compromises as to the thoroughness of my explanations because I was getting too bogged down in them. I eventually had to make the judgement call that the proof should be understandable to the average reader of my article, i.e. college freshmen.

Figure 3: A list of my observations written throughout the process of writing the proof. These are observations about the writing process itself, not about the proof in question.

This approach made the proof as a whole messier and more riddled with confusing digressions that interrupted the flow of the logic presented. I therefore resolved that my approach was flawed and decided that not every little aspect of the proof needed careful explanation. This made the writing process much easier and the final proof easier to read.

This planning process brings us back to CHAT yet again. It is a clear example of representation, as defined by Walker: "The term 'representation' highlights issues related to the way that the people who produce a text conceptualize and plan it" (Walker, p. 75) This also ties rather neatly into distribution, which "involves the consideration of who a text is given to, for what purposes, using what kinds of distribution tools" (Walker, p. 75). As I explained, my first consideration was that my proof be approachable to the mathematical layman, but this mindset was, in the end, unconstructive and counterintuitive. I therefore reasoned that I should construct the proof to be understandable for the average college student, in consideration of the intended audience of this article. This conclusion was perhaps the most

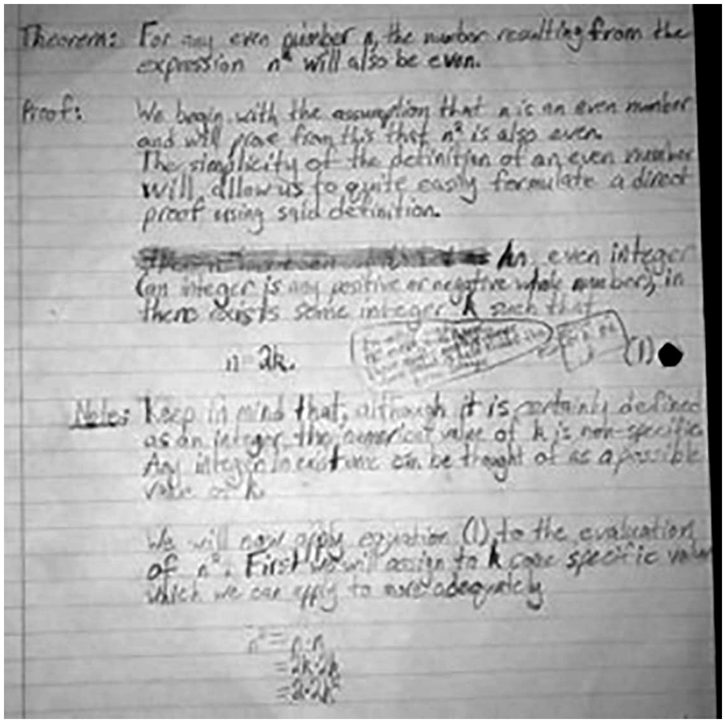


Figure 4: My first attempt at this mathematical proof. As can be seen, this draft was far more focused on explaining every detail, and, as a result, each step in the logic quite simply failed to flow seamlessly into the next.

relevant to the purpose of CHAT because it forced me to consider my proof, not only as a mathematical text but also as a small part of the construction of this article. It therefore helped me to view my writing process in the correct context and, in turn, improved the overall quality of my mathematical writing.

Conclusion

Why did I write this article? A strange question to ask within the article itself, but the answer comes in multiple parts and is very important to the main topic of this journal. It's why I devoted an entire section to the problem of teaching and learning proofs, and it's why I made repeated assertions to their absolute necessity, as well as to the importance of the actual *writing* involved in proof writing. I even included an interview with a professional mathematician to emphasize both of those last two points. And it all boils down to this:

As a mathematics major, I never thought that I'd ever find myself writing something as involved and lengthy as this article. If you had told me, even as recently as the beginning of the semester, that I'd not only write a journal article spanning more than ten pages but that I'd volunteer to do so of my own free will, I'd have laughed in your face. I'm far too much of a perfectionist about my own academic writing to actually enjoy doing it. But, if anything, this only reinforces my point. Because the answer to the question asked at the beginning of this section is that I wanted to teach, to the freshmen who will read this in English 101, the lesson that I wish I had been taught at their age, instead of having to learn it myself as an upperclassman: that, no matter your major or career choice, as much as I'd like for this not to be the case, there is a certain amount of writing that will be required of you. As strange as it may seem, even in a field like mathematics, quality writing skills are essential to your future career. And that leads me to the second part of my answer.

The other reason that I wrote ten-plus pages about mathematical proof writing is to make a point to the freshman math majors who read this. It doesn't matter whether you like them or not; a good deal of the math courses you will take past calculus will require you to familiarize yourself with multiple proofs. Proofs are everywhere in mathematics, and they're really not all that scary. *But I strongly encourage you to not only practice formulating proofs but writing them as well*. The writing of a mathematical proof is like anything else in either writing or math, and indeed in life, in that it gets better and easier with practice.

In a very real way, writing a mathematical proof is more akin to writing an essay than solving a math problem. In fact, since the proof has to be figured out before it can be formally written, the actual writing of a mathematical proof takes place *after* the problem has already been solved. The only thing left to do at that point is to explain the solution in writing. And said writing is no less a part of the field of mathematics than the numbers and equations. Because, as I learned over the course of this entire semester, I was, until now, ignorant of the true definition of mathematics. As Dr. Chebolu explained in my interview with him, mathematics isn't just numbers and calculation: it's the analytical representation of ideas. And it's not complete without the communication and translation of those ideas from numbers and symbols into words. In other words, even mathematicians need to know how to write. And this article is the proof.

Works Cited

Clegg, B. (2011). The Dangerous Ratio. Retrieved from https://nrich.maths.org/2671

Mbaïtiga, Z. (2009). Why College or University Students Hate Proofs in Mathematics. *Journal of Mathematics and Statistics, 5*(1), 32–41. doi:10.3844/jmssp.2009.32.41

Stokdyk, D. (2016). Importance of Mathematics and Why We Study It. Retrieved from https://www.snhu.edu/about-us/news-and-events/2016/08/what-is-a-degree-in-math-and-why-is-it-valuable

Walker, J.R. (2010). Just CHATting. *Grassroots Writing Research Journal, 1*. Retrieved from http://isuwriting.com/wp-content/uploads/2015/04/Walker_Joyce_Just_Chatting.pdf

Appendix

Below is the final version of the proof that I wrote for the purpose of writing this article.

Theorem. The square of any even number is also an even number.

***Proof*:** We assume that a number a is an even number. We will construct a direct proof by showing, given this assumption, that a^2 is also even, using the definition of an even number.

An integer a is even if and only if there exists some other integer k such that

$$a = 2k \tag{1}$$

By squaring both sides of equation (1), we obtain

$$\begin{aligned} a^2 &= (2k)^2 \\ &= 2^2 \times k^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned} \tag{2}$$

Since k is an integer, $2k^2$ must also be an integer. Thus, equation (2) fits the definition of an even number. Therefore, when a is an even number, a^2 must also be an even number.

Special Thanks

I would like to thank the ISU professors without whom I could not have completed this article:

- To Dr. Chebolu: for our very insightful interview.
- To Dr. Zhao: for introducing me to the story of the discovery of irrational numbers.
- And finally, to Dr. Marshall: for encouraging me to pursue this endeavor to completion, for her unwavering belief in my writing ability, and for her flexibility and patience in allowing me the time I needed to write the article to the rather exacting standards that I stubbornly insist on holding myself to.

Jonathan Sabin is a senior at Illinois State University where he previously studied Music Education before changing his major to General Mathematics. He is still a greatly skilled concert trombonist and remains active in the ISU School of Music. After graduating, Sabin intends to pursue a career in voice acting. Oh, he also grudgingly admits to being halfway decent at academic writing.